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1996 J. Phys. A: Math. Gen. 29 L237

(http://iopscience.iop.org/0305-4470/29/10/003)

J. Phys. A: Math. Gen. **29** (1996) L237–L239. Printed in the UK

LETTER TO THE EDITOR

Complementarity and Cirel'son's inequality

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Received 20 October 1995

Abstract. The least upper bound in Cirel'son's quantum generalization of the local-realistic Bell inequality is intermediate between the local-realistic and general probablistic limits. An alternative derivation of Cirel'son's inequality is presented which sheds light on the role played by complementarity in determining this bound and the fact that it falls short of the general probablistic limit.

Perhaps the most celebrated recent contribution to the ongoing debate over how the physical world, as described by quantum theory, should be understood is Bell's theorem [1]. Bell's theorem is a proof that local realism, embodied in certain modest probablistic assumptions, is incompatible with some of the predictions of quantum theory, which have subsequently been corroborated by experiment [2]. It was formulated as a response to the famous argument put forward in 1935 by Einstein, Podolsky and Rosen (EPR) [3] to demonstrate the incompatability of the principles of complementarity (that non-commuting observables cannot simultaneously be predictable 'elements of reality' and locality) which postulates that no pair of spacelike separated events, including the outcomes of measurements and choices regarding what is to be measured, can influence one another.

It follows from the EPR argument that the principle of locality could only be retained at the expense of dispensing with complementarity. Instead, one would have to accept the possibilty that a theory could be constructed in which all observables evolve deterministically, although perhaps depending on some hidden variables.

The importance of Bell's theorem with regard to this conclusion lies in the fact that it exposes a conflict between quantum mechanics and this local-realistic perspective. It provides an experimentally realizable situation in which quantum mechanics and the local-realistic hypothesis *per se* make different numerical predictions. This scenario is as follows. Consider a pair of systems labelled 1 and 2 which have the pairs of observable properties (a, a') and (b, b') respectively. The magnitudes of all observables are bounded as one. We denote the expectation value of the product of the possible results found by measuring a and b by $E(a, b)$ and likewise with other products of two observables, one from each system. If we make the assumption of local reality, then we are led to the conclusion that Bell's inequality

$$|E(a, b) + E(a, b')| + |E(a', b) - E(a', b')| \leqslant 2 \tag{1}$$

must be satisfied.

The predictions of quantum mechanics are not, in general, consistent with this inequality. Indeed, all entangled pure states have been shown by Gisin [4] to lead to a violation of

0305-4470/96/100237+03$19.50

a Bell inequality. As pointed out by Cirel'son [5], quantum systems do obey a similar inequality with a less stringent bound. On making the transition to quantum mechanics, the classical observables (a, a', b, b') are replaced by the corresponding Hermitian operators $(\hat{a}, \hat{a}', \hat{b}, \hat{b}')$. Operators which act on different systems commute. If the eigenvalues of all four quantum observables are ± 1, then we are led to Cirel'son's inequality

$$|E(\hat{a}, \hat{b}) + E(\hat{a}, \hat{b}')| + |E(\hat{a}', \hat{b}) - E(\hat{a}', \hat{b}')| \leqslant 2\sqrt{2}. \tag{2}$$

The quantum mechanical least upper bound given by (2) lies between the local-realistic bound of two set by Bell's inequality and the absolute probablistic limit of four which follows from the fact that the absolute value of each of the expectation values in (2) is less than or equal to one. As shown by Cirel'son [5] and elaborated upon by Braunstein *et al* [6] and Landau [7], the quantum violation of the classical Bell inequality is a consequence of the fact that quantum observables obey a non-commutative algebra. Indeed, the quantum mechanical correction to the local-realistic bound is a homogeneous function of the uncertainty products for non-commuting observables; grounding the observable consequences of non-locality in terms of complementarity.

In previous derivations of Cirel'son's inequality [5–7], the presence of commutators between observables has invariably been recognized as the source of the non-locality implied in (2). While the attribution of the bound in (2) to the incompatability of the measured quantities is quite appropriate and renders formally intelligible the link between non-locality and complementarity, the absolute identification of non-classicity and non-commutativity is complicated by the fact that many observables are, up to an imaginary factor, commutators of other observables. This conceptual difficulty is of particular concern when we have in mind Lie algebraic systems, where it is indeed the rule and of which the spin-$\frac{1}{2}$ arrangement often assumed in discussions of Cirel'son's inequality is an elementary example.

In this short letter, we show that this property of spin operators can be exploited to give a derivation of Cirel'son's inequality where the left-hand side of (2) is expressed solely in terms of Heisenberg uncertainty relations pertaining to a new set of observables. The actual value of this expression is then a measure of the incompatability of these observables, giving them, and the inequality as a whole, a quantum mechanical significance which is not shared by their classical counterparts, and which may open up the possibility of alternative measurement schemes.

Our proof lacks some of the generality of earlier treatments [6, 7]. However, our aim is not the derivation itself but rather to establish a connection with Heisenberg's uncertainty principle, which has hitherto remained concealed owing to the fact that extant derivations place principal emphasis on the difference between Cirel'son's and the Bell–CHSH inequalities rather than the Lie algebraic structure underlying the former.

We make the conventional representation of the systems as spin-$\frac{1}{2}$ particles, and the operator observables $(\hat{a}, \hat{a}', \hat{b}, \hat{b}')$ as single-particle spin operators

$$\hat{a} = \boldsymbol{a} \cdot \boldsymbol{\sigma}^1 \qquad \hat{a}' = \boldsymbol{a}' \cdot \boldsymbol{\sigma}^1 \qquad \hat{b} = \boldsymbol{b} \cdot \boldsymbol{\sigma}^2 \qquad \hat{b}' = \boldsymbol{b}' \cdot \boldsymbol{\sigma}^2 \tag{3}$$

where $\boldsymbol{a}$, $\boldsymbol{a}'$, $\boldsymbol{b}$ and $\boldsymbol{b}'$ are unit vectors and the superscripts 1 and 2 refer to particles 1 and 2, respectively. The Heisenberg uncertainty relation states that the expectation value of the commutator of a pair of operators, $\hat{A}$ and $\hat{B}$, satisfies the inequality $|\langle[\hat{A}, \hat{B}]\rangle| \leqslant 2\Delta A \Delta B$, where ΔA and ΔB are the uncertainties in $\hat{A}$ and $\hat{B}$ respectively. In what follows, we shall define two new pairs of observables, $(\hat{C}, \hat{D})$ and $(\hat{C}', \hat{D}')$, in terms of $\hat{a}$, $\hat{a}'$, $\hat{b}$ and $\hat{b}'$ and show that the sum of the Heisenberg uncertainty relations for these two pairs is equal to Cirel'son's quantum generalization of Bell's inequality, equation (2).

The new observables are defined as

$$\hat{C} = \boldsymbol{\alpha} \cdot \boldsymbol{\sigma}^1 \qquad \hat{C}' = \boldsymbol{\beta} \cdot \boldsymbol{\sigma}^2 \qquad \hat{D} = \boldsymbol{\gamma} \cdot \boldsymbol{\sigma}^1 \boldsymbol{\delta} \cdot \boldsymbol{\sigma}^2 \qquad \hat{D}' = \boldsymbol{\theta} \cdot \boldsymbol{\sigma}^1 \boldsymbol{\eta} \cdot \boldsymbol{\sigma}^2 \tag{4}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ are real vectors. The uncertainties associated with these observables are constrained by the Heisenberg uncertainty relations

$$\Delta C \Delta D \geqslant |\langle(\boldsymbol{\sigma} \times \boldsymbol{\gamma}) \cdot \hat{\boldsymbol{\sigma}}^1 \boldsymbol{\delta} \cdot \hat{\boldsymbol{\sigma}}^2\rangle| \tag{5}$$

$$\Delta C' \Delta D' \geqslant |\langle(\boldsymbol{\beta} \times \boldsymbol{\eta}) \cdot \hat{\boldsymbol{\sigma}}^1 \boldsymbol{\theta} \cdot \hat{\boldsymbol{\sigma}}^2\rangle|. \tag{6}$$

It follows from the definitions in (4) that the maximum uncertainties for the observables $\hat{C}$, $\hat{C}'$, $\hat{D}$ and $\hat{D}'$ are α, β, $\gamma\delta$ and $\theta\eta$, respectively. If we define the Greek vectors in such a way that the following relations hold

$$\boldsymbol{a} + \boldsymbol{a}' = \boldsymbol{\alpha} \times \boldsymbol{\gamma} \qquad \boldsymbol{a} - \boldsymbol{a}' = \boldsymbol{\theta} \qquad \boldsymbol{b} = \boldsymbol{\delta} \qquad \boldsymbol{b}' = \boldsymbol{\beta} \times \boldsymbol{\eta} \tag{7}$$

then we may, without loss of generality, choose $\boldsymbol{\alpha}$ to be orthogonal to $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ to be orthogonal to $\boldsymbol{\eta}$. The easily proven relation follows

$$|\langle(\boldsymbol{\alpha} \times \boldsymbol{\gamma}) \cdot \hat{\boldsymbol{\sigma}}^1 \boldsymbol{\delta} \cdot \hat{\boldsymbol{\sigma}}^2\rangle| + |\langle(\boldsymbol{\beta} \times \boldsymbol{\eta}) \cdot \hat{\boldsymbol{\sigma}}^1 \boldsymbol{\theta} \cdot \hat{\boldsymbol{\sigma}}^2\rangle| \leqslant \alpha\gamma\delta + \beta\eta\theta \tag{8}$$

leading to the inequality

$$|E(\hat{a}, \hat{b}) + E(\hat{a}, \hat{b}')| + |E(\hat{a}', \hat{b}) - E(\hat{a}', \hat{b}')| \leqslant |\boldsymbol{a} + \boldsymbol{a}'| + |\boldsymbol{a} - \boldsymbol{a}'|. \tag{9}$$

Since $\boldsymbol{a}$ and $\boldsymbol{a}'$ are unit vectors, the least upper bound on the right-hand side of (9) is easily shown to be $2\sqrt{2}$, giving Cirel'son's bound. It is clear then that Cirel'son's inequality (2) can be regarded as the sum of two Heisenberg uncertainty relations. Furthermore, it is a consequence of the operators defined in (4) being bounded that the quantum mechanical bound falls short of the general probablistic limit.

The quantum violation of Bell's inequality is a clear demonstration of the fact that no local-realistic theory can reproduce the predictions of quantum theory. The quantum mechanical least upper bound of $2\sqrt{2}$ is, however, less than the absolute probablistic limit of four. This is a consequence of the fact that in quantum mechanics, for a given choice of observables, the least upper bound is equal to the sum of uncertainty products $\Delta C \Delta D + \Delta C' \Delta D'$, having maximum values determined by the bounds on the corresponding operators. It follows from the vector definitions in (7) that this bound is less than the *a priori* limit of four.

AC would like to thank the EPSRC for the award of a research studentship. We are also grateful to Matthew Brownlie for many stimulating discussions.

References

[1] Bell J S 1987 *Speakable and Unspeakable in Quantum Mechanics* (Cambridge: Cambridge University Press)
[2] Clauser J F and Shimony A 1978 *Rep. Prog. Phys.* **4** 1881
[3] Einstein A, Podolsky B and Rosen N 1935 *Phys. Rev.* **47** 777
[4] Gisin N 1991 *Phys. Lett.* **154A** 201
[5] Cirel'son B S 1980 *Lett. Math. Phys.* **4** 93
[6] Braunstein S L, Mann A and Revsen M 1992 *Phys. Rev. Lett.* **68** 3259
[7] Landau L J 1987 *Phys. Lett.* **120A** 54